

# Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables

Valeri P. Frolov and David Kubizňák<sup>1</sup>

<sup>1</sup> *Theoretical Physics Institute, University of Alberta,  
Edmonton, Alberta, Canada T6G 2G7 \**

In this paper, we discuss hidden symmetries in rotating black hole spacetimes. We start with an extended introduction which mainly summarizes results on hidden symmetries in four dimensions and introduces Killing and Killing–Yano tensors, objects responsible for hidden symmetries. We also demonstrate how starting with a principal CKY tensor (that is a closed non-degenerate conformal Killing–Yano 2-form) in 4D flat spacetime one can ‘generate’ 4D Kerr-NUT-(A)dS solution and its hidden symmetries. After this we consider higher-dimensional Kerr-NUT-(A)dS metrics and demonstrate that they possess a principal CKY tensor which allows one to generate the whole tower of Killing–Yano and Killing tensors. These symmetries imply complete integrability of geodesic equations and complete separation of variables for the Hamilton–Jacobi, Klein–Gordon, and Dirac equations in the general Kerr-NUT-(A)dS metrics.

PACS numbers: 04.50.Gh, 04.50.-h, 04.70.-s, 04.20.Jb

## I. INTRODUCTION

### A. Symmetries

In modern theoretical physics one can hardly overestimate the role of symmetries. They comprise the most fundamental laws of nature, they allow us to classify solutions, in their presence complicated physical problems become tractable. The value of symmetries is especially high in nonlinear theories, such as general relativity.

---

\*Electronic address: frolov@phys.ualberta.ca, kubiznak@phys.ualberta.ca

In curved spacetime continuous symmetries (isometries) are generated by Killing vector fields. Such symmetries have clear geometrical meaning. Let us assume that in a given manifold we have a 1-parameter family of diffeomorphisms generated by a vector field  $\xi$ . Such a vector field determines the dragging of tensors by the diffeomorphism transformation. If a tensor field  $T$  is invariant with respect to this dragging, that is its Lie derivative along  $\xi$  vanishes,  $\mathcal{L}_\xi T = 0$ , we have a symmetry. A vector field which generates transformations preserving the metric is called a *Killing vector field*, and the corresponding diffeomorphism—an *isometry*. For each of the Killing vector fields there exists a conserved quantity. For example, for a particle geodesic motion this conserved quantity is a projection of the particle momentum on the Killing vector.

Besides isometries the spacetime may also possess *hidden symmetries*, generated by either symmetric or antisymmetric tensor fields. Such symmetries are not directly related to the metric invariance under diffeomorphism transformations. They represent the genuine symmetries of the phase space rather than the configuration space. For example, symmetric Killing tensors give rise to conserved quantities of higher order in particle momenta, and underline the separability of scalar field equations. Less known but even more fundamental are antisymmetric Killing–Yano tensors which are related to the separability of field equations with spin, the existence of quantum symmetry operators, and the presence of conserved charges.

## B. Miraculous properties of the Kerr geometry

To illustrate the role of hidden symmetries in general relativity let us recapitulate the “miraculous” properties [1] of the Kerr geometry. This astrophysically important metric was obtained in 1963 by Kerr [2] as a special solution which can be presented in the Kerr–Schild form [3]

$$g_{ab} = \eta_{ab} + 2Hl_a l_b, \quad (1)$$

where  $\boldsymbol{\eta}$  is a flat metric and  $\boldsymbol{l}$  is a null vector, in both metrics  $\boldsymbol{g}$  and  $\boldsymbol{\eta}$ .<sup>1</sup> The Kerr solution is stationary and axially symmetric, and it belongs to the metrics of the special algebraic type **D** of Petrov's classification [6].

Although the Killing vector fields  $\partial_t$  and  $\partial_\phi$  are not enough to provide a sufficient number of integrals of motion<sup>2</sup> in 1968 Carter [7, 8] demonstrated that both—the Hamilton–Jacobi and scalar field equations—can be separated. This proved, apart from other things, that there exists an additional integral of motion, ‘mysterious’ Carter’s constant, which makes the particle geodesic motion completely integrable. In 1970, Walker and Penrose [9] pointed out that Carter’s constant is quadratic in particle momenta and its existence is directly connected with the symmetric Killing tensor [10]

$$K_{ab} = K_{(ab)}, \quad K_{(ab;c)} = 0. \quad (2)$$

During the several following years it was discovered that it is not only the Klein–Gordon equation which allows the separation of variables in the Kerr geometry. In 1972, Teukolsky decoupled the equations for electromagnetic and gravitational perturbations, and separated variables in the resulting master equations [11]. One year later the massless neutrino equation by Teukolsky and Unruh [12, 13], and in 1976 the massive Dirac equation by Chandrasekhar and Page [14, 15] were separated.

Meanwhile a new breakthrough was achieved in the field of hidden symmetries when in 1973 Penrose and Floyd [16] discovered that the Killing tensor for the Kerr metric can be written in the form

$$K_{ab} = f_{ac} f_b^c, \quad (3)$$

where the antisymmetric tensor  $\boldsymbol{f}$  is the Killing–Yano (KY) tensor [17]

$$f_{ab} = f_{[ab]}, \quad f_{a(b;c)} = 0. \quad (4)$$

---

<sup>1</sup> If the ansatz (1) is inserted into the Einstein equations, one effectively reduces the problem to a linear one (see, e.g., [4]). This gives a powerful tool for the study of special solutions of the Einstein equations. This method works in higher dimensions as well. For example, the Kerr–Schild ansatz was used by Myers and Perry to obtain their higher-dimensional black hole solutions [5].

<sup>2</sup> For example, for a particle motion these isometries generate the conserved energy and azimuthal component of the angular momentum, which together with the conservation of  $p^2$  gives only 3 integrals of motion. For separability of the Hamilton–Jacobi equation in the Kerr spacetime the fourth integral of motion is required.

A Killing–Yano tensor is in many aspects more fundamental than a Killing tensor. Namely, its ‘square’ is always Killing tensor, but in a general case, the opposite is not true [18].

Many of the remarkable properties of the Kerr spacetime are consequences of the existence of the Killing–Yano tensor. In particular, in 1974 Collinson demonstrated that the integrability conditions for a non-degenerate Killing–Yano tensor imply that the spacetime is necessary of the Petrov type **D** [19].<sup>3</sup> In 1975, Hughston and Sommers showed that in the Kerr geometry the Killing–Yano tensor  $\mathbf{f}$  generates both of its isometries [22]. Namely, the Killing vectors  $\boldsymbol{\xi}$  and  $\boldsymbol{\eta}$ , generating the time translation and the rotation, can be written as follows:

$$\xi^a = \frac{1}{3}(*f)^{ba}{}_{;b} = (\partial_t)^a, \quad \eta^a = -K^a_b \xi^b = (\partial_\phi)^b. \quad (5)$$

This in fact means that all the symmetries necessary for complete integrability of geodesic motion are ‘derivable’ from the existence of this Killing–Yano tensor.

In 1977, Carter demonstrated [23] that given an isometry  $\boldsymbol{\xi}$  and/or a Killing tensor  $\mathbf{K}$  one can construct the operators

$$\hat{\xi} = i\xi^a \nabla_a, \quad \hat{K} = \nabla_a K^{ab} \nabla_b, \quad (6)$$

which commute with the scalar Laplacian<sup>4</sup>

$$[\square, \hat{\xi}] = 0 = [\square, \hat{K}], \quad \square = \nabla_a g^{ab} \nabla_b. \quad (7)$$

Under additional conditions, satisfied by the Kerr geometry, these operators commute also between themselves and provide therefore good quantum numbers for scalar fields. In 1979 Carter and McLenaghan found that an operator

$$\hat{f} = i\gamma_5 \gamma^a \left( f_a{}^b \nabla_b - \frac{1}{6} \gamma^b \gamma^c f_{ab;c} \right) \quad (8)$$

commutes with the Dirac operator  $\gamma^a \nabla_a$  [25]. This gives a new quantum number for the spinor wavefunction and explains why separation of the Dirac equation can be achieved.

---

<sup>3</sup> All the vacuum type **D** solutions were obtained by Kinnersley [20]. Demianski and Francaviglia showed that in the absence of acceleration these solutions admit Killing and Killing–Yano tensors [21].

<sup>4</sup> In fact, the operator  $\hat{K}$  defined by (6) commutes with  $\square$  provided that the background metric satisfies the vacuum Einstein or source-free Einstein–Maxwell equations. In more general spacetimes, however, a quantum anomaly proportional to a contraction of  $\mathbf{K}$  with the Ricci tensor may appear. Such anomaly is not present if an additional condition (3) is satisfied [24].

Similar symmetry operators for other equations with spin, including electromagnetic and gravitational perturbations, were constructed later [26].

In 1983, Marck solved equations for the parallel transport of an orthonormal frame along geodesics in the Kerr spacetime and used this result for the study of tidal forces [27]. For this construction he used a simple fact that the vector

$$L_a = f_{ab}p^b, \quad L_ap^a = 0, \quad (9)$$

is parallel-propagated along a geodesic  $\mathbf{p}$ .

In 1987, Carter [28] pointed out that the Killing–Yano tensor itself is derivable from a 1-form  $\mathbf{b}$ ,

$$\mathbf{f} = *\mathbf{db}. \quad (10)$$

Such a form  $\mathbf{b}$  is usually called a KY *potential*. It satisfies the Maxwell equations and can be interpreted as a 4-potential of an electromagnetic field with the source current proportional to the *primary* Killing vector field  $\boldsymbol{\partial}_t$ , cf. eq. (5). In the paper [29], the equations for an equilibrium configuration of a cosmic string near the Kerr black hole were separated. In 1993, Gibbons *et al.* demonstrated that due to the presence of Killing–Yano tensor the classical spinning particles in this background possess enhanced worldline supersymmetry [30]. Conserved quantities in the Kerr geometry generated by  $\mathbf{f}$  were discussed in 2006 by Jezierski and Łukasik [31].

To conclude this section we mention that many of the above statements and results, which we have formulated for the Kerr geometry, are in fact more general. Their validity can be extended to more general spacetimes, or even to an arbitrary number of spacetime dimensions. For example, the whole Carter’s class of solutions [32] admits a KY tensor and possesses many of the discussed properties.<sup>5</sup> General results on Killing–Yano tensors and algebraic properties were gathered by Hall [34]. A relationship among the existence of Killing tensors, Killing–Yano tensors, and separability structures for the Hamilton–Jacobi equation in arbitrary number of spacetime dimensions was discussed in [21, 35].

---

<sup>5</sup> A general form of a line element in four dimensions admitting a Killing–Yano tensor was obtained by Dietz and Rüdiger [33].

### C. Higher-dimensional black holes

Recently, a lot of interest focuses on higher-dimensional ( $D > 4$ ) black hole spacetimes. In the widely discussed models with large extra dimensions it is assumed that one or more additional spatial dimensions are present. In such models one expects mini black hole production in the high energy particle collisions [36]. Mini black holes can serve as a probe of the extra dimensions. At the same time their interaction with the brane, representing our physical world, can give the information about the brane properties. If a black hole is much smaller than the size of extra dimension and the brane tension is neglected, its metric can be approximated by an asymptotically flat or (A)dS solution of the higher-dimensional Einstein equations.

Study of higher-dimensional black hole solutions has a long history. In 1963, Tangherlini [37] obtained a higher-dimensional generalization of the Schwarzschild metric [38]. The charged version of the Tangherlini metric was found in 1986 by Myers and Perry [5]. In the same paper a general vacuum rotating black hole in higher dimensions was obtained. This solution, often called the Myers–Perry (MP) metric, generalizes the four-dimensional Kerr solution. Main new feature of the MP metrics in  $D$  dimensions is that instead of 1 rotation parameter, they have  $[(D-1)/2]$  rotation parameters, corresponding to  $[(D-1)/2]$  independent 2-planes of rotation.

Later, in 1998, Hawking, Hunter, and Taylor–Robinson [39] found a 5D generalization of the 4D rotating black hole in asymptotically (anti) de Sitter space (Kerr-(A)dS metric [8]). In 2004 Gibbons, Lü, Page, and Pope [40, 41] discovered the general Kerr-de Sitter metrics in arbitrary number of dimensions. After several attempts to include NUT [42] parameters [43, 44], in 2006 Chen, Lü, and Pope [45] found a general Kerr-NUT-(A)dS solution of the Einstein equations for all  $D$ .

These metrics were obtained in special coordinates which are the natural higher-dimensional generalization of the Carter’s 4D *canonical coordinates* [32]. So far this metric remains the most general black hole type solution of the Einstein equations with the cosmological constant (with horizon of the spherical topology) which is known analytically.<sup>6</sup> For

---

<sup>6</sup> Besides the brane-world scenarios, these black holes find their applications in the ADS/CFT correspondence. In the BPS limit odd dimensional metrics lead to the Sasaki–Einstein metrics [45, 46] whereas even dimensional metrics lead to the Calabi–Yau spaces [47]. There have been also several attempts to

a recent extended review on higher-dimensional black holes see [51].

In connection with these black holes a natural question arises: To what extent are the remarkable properties of four-dimensional black holes carried by their higher-dimensional analogues? And in particular, do these spacetimes possess hidden symmetries?

The hidden symmetries of higher-dimensional rotating black holes were first discovered for the 5D Myers–Perry metrics [52, 53]. It was demonstrated that both, the Hamilton–Jacobi and scalar field equations, allow the separation of variables and the corresponding Killing tensor was obtained. This, for example, allows one to obtain a cross-section for the capture of particles and light by 5D black holes [54]. Later it was shown that 5D results can be extended to arbitrary number of dimensions, provided that rotation parameters of the MP metric can be divided into two classes, and within each of the classes these parameters are equal one to another. Similar results were found in the presence of the cosmological constant and NUT parameters [43, 55]. It was also demonstrated that a stationary string configuration in the 5D Myers–Perry spacetime is completely integrable [56].

#### D. Recent developments

Recently, a new breakthrough in the study of higher-dimensional rotating black holes was achieved. It turned out that the properties of even the most general known higher-dimensional Kerr–NUT–(A)dS black holes [45] are, in many aspects, similar to the properties of their 4-dimensional ‘cousins’. These follow from the existence of a special closed conformal Killing–Yano (CKY) tensor, which is called *principal*.

The principal CKY tensor was first discovered for the Myers–Perry metrics [57], and soon after that for the completely general Kerr–NUT–(A)dS spacetimes [58]. Starting with this tensor, one can generate the whole *tower* of Killing–Yano and Killing tensors [59] which are responsible for complete integrability of geodesic motion in these spacetimes [60, 61]. Such integrability was independently proved by separating the Hamilton–Jacobi and Klein–Gordon equations [62]. These results are very promising and might suggest that also the equations with spin can be decoupled and separated. The separation of the Dirac equation was already demonstrated [63]. Also worth mentioning is a recent work on separability of

---

generalize these solutions. For example, to find a similar solution of the Einstein–Maxwell equations either in an analytical form [48] or numerically [49]. See also [50].

vector and tensor fields in  $D = 5$  Myers–Perry spacetimes with equal angular momenta [64], and proved complete integrability of stationary string configurations in general Kerr-NUT-(A)dS spacetimes [65]. For a brief review of these results see [66].

In this paper, we discuss the hidden symmetries of rotating black holes. The following section contains basic definitions. In Section III, starting from a 4D flat space and choosing a special principal CKY tensor, we ‘derive’ the 4D Kerr-NUT-(A)dS spacetime and its hidden symmetries. In Section IV we prove the central theorem concerning the properties of closed CKY tensors and introduce the tower of Killing and Killing–Yano tensors in higher-dimensional spacetimes. We also construct the canonical basis associated with the principal CKY tensor. In Section V we apply these results to higher-dimensional Kerr-NUT-(A)dS solutions. Section VI contains discussion of the separability problem. Possible future developments are discussed in Section VII.

## II. KILLING–YANO AND KILLING TENSORS

### A. Definitions

Let us consider a  $D$ -dimensional spacetime with a metric  $\mathbf{g}$ . In order to simultaneously cover both cases of odd and even dimensions we write  $D = 2n + \varepsilon$ , where  $\varepsilon = 0$  ( $\varepsilon = 1$ ) for the even (odd) dimensional case. A spacetime possesses an isometry generated by the Killing vector field  $\boldsymbol{\xi}$  if this vector obeys the *Killing equation*

$$\xi_{(a;b)} = 0. \quad (11)$$

For a geodesic motion of a particle in such curved spacetime the quantity  $p^a \xi_a$ , where  $\mathbf{p}$  is the momentum of the particle, remains constant along the particle’s trajectory. Similarly, for a null geodesic,  $p^a \xi_a$  is conserved provided  $\boldsymbol{\xi}$  is a *conformal Killing vector* obeying the equation

$$\xi_{(a;b)} = \tilde{\xi} g_{ab}, \quad \tilde{\xi} = D^{-1} \xi^b{}_{;b}. \quad (12)$$

There exist two natural (symmetric and antisymmetric) generalizations of a (conformal) Killing vector.

A symmetric (rank- $p$ ) *conformal Killing tensor* [9]  $\mathbf{K}$  obeys the equations

$$K_{a_1 a_2 \dots a_p} = K_{(a_1 a_2 \dots a_p)}, \quad K_{(a_1 a_2 \dots a_p; b)} = g_{b(a_1} \tilde{K}_{a_2 \dots a_p)}. \quad (13)$$



As in the case of a conformal Killing vector, the tensor  $\tilde{\mathbf{K}}$  is determined by tracing both sides of equation (13). If  $\tilde{\mathbf{K}}$  vanishes, the tensor  $\mathbf{K}$  is called a *Killing tensor* [10]. In a presence of the Killing tensor  $\mathbf{K}$  the conserved quantity for a geodesic motion is

$$K = K_{a_1 a_2 \dots a_p} p^{a_1} p^{a_2} \dots p^{a_p} . \quad (14)$$

For null geodesics this quantity is conserved not only for a Killing tensor, but also for a conformal Killing tensor.

A *conformal Killing-Yano* (CKY) tensor [24, 67]  $\mathbf{h}$  is an *antisymmetric* tensor

$$h_{a_1 a_2 \dots a_p} = h_{[a_1 a_2 \dots a_p]} \quad (15)$$

which obeys

$$\nabla_{(a_1} h_{a_2) a_3 \dots a_{p+1}} = g_{a_1 a_2} \tilde{h}_{a_3 \dots a_{p+1}} - (p-1) g_{[a_3 (a_1} \tilde{h}_{a_2) \dots a_{p+1}]} . \quad (16)$$

By tracing both sides of this equation one obtains the following expression for  $\tilde{\mathbf{h}}$

$$\tilde{h}_{a_2 a_3 \dots a_p} = \frac{1}{D-p+1} \nabla^{a_1} h_{a_1 a_2 \dots a_p} . \quad (17)$$

In the case when  $\tilde{\mathbf{h}} = 0$  one has a *Killing-Yano* (KY) tensor [17]. For the KY tensor  $\mathbf{h}$  the quantity

$$L_{a_1 a_2 \dots a_{(p-1)}} = h_{a_1 a_2 \dots a_p} p^{a_p} , \quad (18)$$

is parallel-propagated along the geodesic  $\mathbf{p}$ .

Let us mention two additional important properties: having a KY tensor  $\mathbf{h}$  the quantity

$$K_{ab} = \frac{c_p}{(p-1)!} h_{aa_2 \dots a_p} h_b^{a_2 \dots a_p} \quad (19)$$

is an associated Killing tensor. Here  $c_p$  is an arbitrary constant, which is often taken to be one. For a different convenient choice see Section IV. For a CKY tensor  $\mathbf{h}$  of rank-2 the vector

$$\xi^{(0)a} = \frac{1}{D-1} \nabla_b h^{ab} \quad (20)$$

obeys the following equation [68]

$$\xi^{(0)}_{(a;b)} = -\frac{1}{D-2} R_{c(a} h_{b)}^c . \quad (21)$$

Thus, in an Einstein space, that is when  $R_{ab} = \Lambda g_{ab}$ ,  $\xi^{(0)}$  is the Killing vector.

### B. Killing–Yano equations in terms of differential forms

The CKY tensors are forms and operations with them are greatly simplified if one uses the ‘language’ of differential forms.

Let us remind some of the relations we shall use later. If  $\alpha_p$  and  $\beta_q$  are  $p$ - and  $q$ -forms, respectively, the external derivative  $d$  of their exterior product  $\wedge$  obeys a relation

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \quad (22)$$

A Hodge dual  $*\alpha_p$  of the  $p$ -form  $\alpha_p$  is  $(D - p)$ -form defined as

$$(*\alpha_p)_{a_1 \dots a_{D-p}} = \frac{1}{p!} \alpha^{b_1 \dots b_p} e_{b_1 \dots b_p a_1 \dots a_{D-p}}, \quad (23)$$

where  $e_{a_1 \dots a_D}$  is a totally antisymmetric tensor. The *co-derivative*  $\delta$  is defined as follows:

$$\delta\alpha_p = (-1)^p \epsilon_p * d * \alpha_p, \quad \epsilon_p = (-1)^{p(D-p)} \frac{\det(g)}{|\det(g)|}. \quad (24)$$

One also has  $**\alpha_p = \epsilon_p \alpha_p$ .

If  $\{e_a\}$  is a basis of vectors, then dual basis of 1-forms  $\omega^a$  is defined by the relations  $\omega^a(e_b) = \delta_b^a$ . We denote  $\eta_{ab} = g(e_a, e_b)$  and by  $\eta^{ab}$  the inverse matrix. The operations with the indices enumerating the basic vectors and forms are performed by using these matrices. In particular,  $e^a = \eta^{ab} e_b$ , and so on. We denote a covariant derivative along the vector  $e_a$  by  $\nabla_a = \nabla_{e_a}$ . One has

$$d = \omega^a \wedge \nabla_a, \quad \delta = -e^a \lrcorner \nabla_a. \quad (25)$$

In tensor notations the ‘hook’ operator (inner derivative) along a vector  $\mathbf{X}$ , applied to a  $p$ -form  $\alpha_p$ , corresponds to a contraction

$$(X \lrcorner \alpha_p)_{a_2 \dots a_p} = X^{a_1} (\alpha_p)_{a_1 a_2 \dots a_p}. \quad (26)$$

It satisfies the properties

$$e^a \lrcorner (\alpha_p \wedge \beta_q) = (e^a \lrcorner \alpha_p) \wedge \beta_q + (-1)^p \alpha_p \wedge (e^a \lrcorner \beta_q), \quad (27)$$

$$e^a \lrcorner \omega_a = D, \quad \omega_a \wedge (e^a \lrcorner \alpha_p) = p \alpha_p. \quad (28)$$

For a given vector  $\mathbf{X}$  one defines  $\mathbf{X}^\flat$  as a corresponding 1-form with the components  $(X^\flat)_a = g_{ab} X^b$ . In particular, one has  $\eta^{ab} (e_b)^\flat = \omega^a$ . We refer to [69, 70] where these and many other useful relations can be found.

Definition (16) of the CKY tensor  $\mathbf{h}$  (which is a  $p$ -form) is equivalent to the following equation [70]:

$$\nabla_X \mathbf{h} = \frac{1}{p+1} \mathbf{X} \lrcorner d\mathbf{h} - \frac{1}{D-p+1} \mathbf{X}^\flat \wedge \delta \mathbf{h}. \quad (29)$$

That is, a CKY tensor is a form for which the covariant derivative splits into the exterior and divergence parts.<sup>7</sup> Using the relation

$$\mathbf{X} \lrcorner * \boldsymbol{\omega} = *(\boldsymbol{\omega} \wedge \mathbf{X}^\flat), \quad (30)$$

it is easy to show that under the Hodge duality the exterior part transforms into the divergence part and vice versa. In particular, (29) implies

$$\nabla_X (*\mathbf{h}) = \frac{1}{p_*+1} \mathbf{X} \lrcorner d(*\mathbf{h}) - \frac{1}{D-p_*+1} \mathbf{X}^\flat \wedge \delta(*\mathbf{h}), \quad p_* = D - p. \quad (31)$$

The Hodge dual  $*\mathbf{h}$  of a CKY tensor  $\mathbf{h}$  is again a CKY tensor.

Two special subclasses of CKY tensors are of particular interest: (a) Killing–Yano tensors with zero divergence part  $\delta \mathbf{h} = 0$  and (b) closed CKY tensors with vanishing exterior part  $d\mathbf{h} = 0$ . Under the Hodge duality these subclasses transform into each other.

For a closed CKY tensor there exists locally a (KY) potential  $\mathbf{b}$ , which is a  $(p-1)$ -form, such that

$$\mathbf{h} = d\mathbf{b}. \quad (32)$$

The Hodge dual of such a tensor  $\mathbf{h}$ ,

$$\mathbf{f} = *\mathbf{h} = *d\mathbf{b}, \quad (33)$$

is a Killing–Yano tensor, cf. eq. (10).

### III. 4D KERR-NUT-(A)DS SPACETIME AND ITS HIDDEN SYMMETRIES

Before we proceed to higher-dimensional rotating black holes and their hidden symmetries it is instructive to illustrate the basic ideas on the well known 4D case. As we shall see later, a key object of the theory in higher dimensions is a *principal CKY tensor*. We start discussing this object and its properties in 4D flat spacetime and demonstrate how it generates other

---

<sup>7</sup> For a general form an additional term, the harmonic part, is present. It is the lack of this term what makes CKY tensors ‘special’.

objects (Killing–Yano and Killing tensors) responsible for the hidden symmetries. We also show how this principal CKY tensor allows one easily to ‘generate’ the 4D Kerr-NUT-(A)dS metric starting from the flat one—written in the *canonical coordinates* determined by this tensor.<sup>8</sup> We also demonstrate the separation of variables in the 4D Kerr-NUT-(A)dS spacetime in the canonical coordinates. It should be emphasized, that this section plays the role of an introduction for newcomers to the field which should illuminate the main ideas of more complicated higher-dimensional theory.

### A. Principal conformal Killing–Yano tensor

Consider a four-dimensional flat spacetime with the metric

$$dS^2 = \eta_{ab}dX^a dX^b = -dT^2 + dX^2 + dY^2 + dZ^2. \quad (34)$$

The principal CKY tensor  $\mathbf{h}$  is the rank-2 closed CKY tensor. Therefore, there exists a 1-form potential  $\mathbf{b}$  so that  $\mathbf{h} = d\mathbf{b}$ . Let us consider the following ansatz:

$$\mathbf{b} = \frac{1}{2} [-R^2 dT + a(Y dX - X dY)], \quad R^2 = X^2 + Y^2 + Z^2. \quad (35)$$

Our choice of this special form for the potential  $\mathbf{b}$  will become clear later, when it will be shown that this is a flat spacetime limit of the potential for the principal CKY tensor in the Kerr-NUT-(A)dS spacetime. For a moment we just mention that the form (35) of the potential  $\mathbf{b}$  singles out time coordinate  $T$ , a two-dimensional  $(X, Y)$  plane in space, and contains an arbitrary constant  $a$ .

It can be easily shown that

$$\mathbf{h} = d\mathbf{b} = dT \wedge (X dX + Y dY + Z dZ) + a dY \wedge dX \quad (36)$$

is a closed conformal Killing–Yano tensor. It means that its dual 2-form  $\mathbf{f} = *\mathbf{h}$  is the Killing–Yano tensor.<sup>9</sup>

$$\mathbf{f} = X dZ \wedge dY + Z dY \wedge dX + Y dX \wedge dZ + a dZ \wedge dT. \quad (38)$$

---

<sup>8</sup> For an alternative ‘derivation’ of the Kerr-NUT-(A)dS spacetime see, e.g., [71] or [72].

<sup>9</sup> In  $D$  dimensions the maximum number of (linear independent) Killing–Yano tensors of a given rank- $p$  is

$$N_p = \binom{D}{p} + \binom{D}{p+1} = \frac{(D+1)!}{(D-p)!(p+1)!}. \quad (37)$$

This reflects the fact that, similar to Killing vectors, Killing–Yano tensors are completely determined by

Let us put, for a moment,  $a = 0$ . Then the KY tensor  $f_{ab}$  has only spatial components  $f_{ik}$ , and the Killing tensor (19), associated with it, reads

$$K_{ij} = R^2 \delta_{ij} - X^i X^j = \sum_{k=X,Y,Z} \xi_{(k)i} \xi_{(k)j}, \quad \xi_{(k)i} = \epsilon_{kji} X^j. \quad (39)$$

Here  $\xi_{(k)i}$  are the spatial rotational Killing vectors. Therefore, the Killing tensor  $\mathbf{K}$  can be written as a sum of products of Killing vectors, and thus it is *reducible*. Parallel-propagated vector (18)

$$L_i = f_{ik} p^k = \epsilon_{ijk} X^j p^k = \xi_{(k)i} p^k \quad (40)$$

has the meaning of the conserved angular momentum.<sup>10</sup> The conserved quantity

$$K(a=0) = \sum_{k=X,Y,Z} L_k^2 = \vec{L}^2 \quad (41)$$

is the square of the total angular momentum.

For  $a \neq 0$  the conserved quantity

$$K = \vec{L}^2 + 2aEL_Z + a^2(E^2 - p_Z^2), \quad (42)$$

is also reducible. Here  $E = -p_T$  and  $p_Z$  are the conserved energy and the momentum in the  $Z$ -direction, respectively.

## B. ‘Derivation’ of the 4D Kerr-NUT-(A)dS metric

Consider a general case with  $a \neq 0$ . We first introduce the ellipsoidal coordinates<sup>11</sup>

$$X = \sqrt{r^2 + a^2} \sin \theta \cos \phi, \quad Y = \sqrt{r^2 + a^2} \sin \theta \sin \phi, \quad Z = r \cos \theta, \quad (43)$$

---

the values of their components and the values of their (completely antisymmetric) first derivatives at a given point. Flat space has the maximum number of independent Killing–Yano tensors of each rank. Any KY tensor there can be written as a linear combination of ‘translational’ KY tensors (which are a simple wedge product of translational Killing vectors) and ‘rotational’ KY tensors (which are a wedge product of translations with a spacetime rotation, completely antisymmetrized) [73]. In particular case of  $D = 4$  we have 10 rank-2 KY tensors (6 translational and 4 rotational).

<sup>10</sup> In general, for a simple spacelike ( $f_{ab}f^{ab} > 0$ ) Killing–Yano tensor  $\mathbf{f}$ , there exists a close analogy between the angular momentum of classical mechanics and the vector  $L^a = f^{ab}p_b$  [33].

<sup>11</sup> In this step, we associate constant  $a$  with ‘rotation’ parameter.

and rewrite the metric, the potential, the principal CKY tensor, and the KY tensor as

$$\begin{aligned}
dS^2 &= -dT^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + a^2 \cos^2 \theta) \left( \frac{dr^2}{r^2 + a^2} + d\theta^2 \right), \\
\mathbf{b} &= \frac{1}{2} \left[ -(r^2 + a^2 \sin^2 \theta) \mathbf{d}T - a \sin^2 \theta (r^2 + a^2) \mathbf{d}\phi \right], \\
\mathbf{h} &= -r \mathbf{d}r \wedge (\mathbf{d}T + a \sin^2 \theta \mathbf{d}\phi) - a \sin \theta \cos \theta \mathbf{d}\theta \wedge [a \mathbf{d}T + (r^2 + a^2) \mathbf{d}\phi], \\
\mathbf{f} &= a \cos \theta \mathbf{d}r \wedge (\mathbf{d}T + a \sin^2 \theta \mathbf{d}\phi) - r \sin \theta \mathbf{d}\theta \wedge [a \mathbf{d}T + (r^2 + a^2) \mathbf{d}\phi].
\end{aligned} \tag{44}$$

Second, we introduce the new coordinates

$$y = a \cos \theta, \quad t = T + a\phi, \quad \psi = -\phi/a, \tag{45}$$

in which the metric takes the ‘algebraic’ form

$$dS^2 = -\frac{\Delta_r (dt + y^2 d\psi)^2}{r^2 + y^2} + \frac{\Delta_y (dt - r^2 d\psi)^2}{r^2 + y^2} + \frac{(r^2 + y^2) dr^2}{\Delta_r} + \frac{(r^2 + y^2) dy^2}{\Delta_y}, \tag{46}$$

where

$$\Delta_r = r^2 + a^2, \quad \Delta_y = a^2 - y^2. \tag{47}$$

The hidden symmetries are

$$\begin{aligned}
\mathbf{b} &= \frac{1}{2} \left[ (y^2 - r^2 - a^2) \mathbf{d}t - r^2 y^2 \mathbf{d}\psi \right], \\
\mathbf{h} &= y \mathbf{d}y \wedge (\mathbf{d}t - r^2 \mathbf{d}\psi) - r \mathbf{d}r \wedge (\mathbf{d}t + y^2 \mathbf{d}\psi), \\
\mathbf{f} &= r \mathbf{d}y \wedge (\mathbf{d}t - r^2 \mathbf{d}\psi) + y \mathbf{d}r \wedge (\mathbf{d}t + y^2 \mathbf{d}\psi).
\end{aligned} \tag{48}$$

In the potential  $\mathbf{b}$  the term proportional to  $a^2$  is constant and may be omitted. We remind that (46)–(47) is just a metric of the flat space written in special coordinates.

Let us consider now the metric (46) without imposing conditions (47) on functions  $\Delta_r$  and  $\Delta_y$ , but assuming that they are functions of  $r$  and  $y$ , respectively. Then substituting this ansatz into the Einstein equations

$$R_{ab} = -3\lambda g_{ab}, \tag{49}$$

one finds that these equations are satisfied provided the following relation is valid:

$$\frac{d^2 \Delta_r}{dr^2} + \frac{d^2 \Delta_y}{dy^2} = 12\lambda(r^2 + y^2). \tag{50}$$

The most general solution of this equation is

$$\Delta_r = (r^2 + a^2)(1 + \lambda r^2) - 2Mr, \quad \Delta_y = (a^2 - y^2)(1 - \lambda y^2) + 2Ny. \tag{51}$$

In other words, a simple replacement of functions (47) by more general polynomials (51) generates a non-trivial solution of the Einstein equations from a flat one. This solution is the Kerr-NUT-(A)dS metric written in the *canonical form* [32]. It obeys the Einstein equations with the cosmological constant (see (49)).  $M$  stands for mass, and parameters  $a$  and  $N$  are connected with rotation and NUT parameter [74].

A remarkable fact is that in canonical coordinates, with arbitrary  $\Delta_r(r)$  and  $\Delta_y(y)$ , the objects  $\mathbf{b}$ ,  $\mathbf{h}$ , and  $\mathbf{f}$  (48) are again the potential, the principal CKY tensor, and the KY tensor for the metric (46) [58, 75]. In particular, these relations give a principal CKY tensor, and a derived from it 2-form of the KY tensor, for the Kerr-NUT-(A)dS spacetime (46), (51).

Let us emphasize that in the Kerr-NUT-(A)dS spacetime neither the square of the total angular momentum,  $\vec{L}^2$ , nor the projection of the momentum on the  $Z$ -axis,  $p_Z$ , which enter (42) have well defined meaning. However, the quadratic in momentum quantity,  $K^{ab}p_ap_b$ , where  $K_{ab} = f_{ac}f_b{}^c$ , is well defined and conserved. In the absence of the cosmological constant and NUT parameter, that is for the Kerr black hole, this quantity can be presented in the form (42) in the asymptotic region, where the spacetime is practically flat. The angular momentum and other quantities which enter (42) must be then understood as the corresponding asymptotically conserved quantities. Since the energy  $E$  and the angular momentum along the axis of symmetry  $L_Z$  are conserved exactly in any stationary axisymmetric spacetime they can be excluded from (42) and the asymptotically conserved quantity can be written as follows [76]:

$$Q = L_X^2 + L_Y^2 - a^2 p_Z^2. \quad (52)$$

For a scattering of particles in the Kerr metric, the presence of an exact integral of motion connected with the Carter's constant implies that the quantity  $Q$  calculated for the incoming from infinity particle must be the same as  $Q$  calculated at the infinity for the outgoing particle. An interesting question is the following: suppose that such a conservation law is established for any scattering of particles by a localized object, can one conclude that the metric of this object possesses a hidden symmetry?

### C. Symmetric form of the metric

Let us perform the ‘Wick’ rotation in radial coordinate  $r$ . This transforms the metric (46) and its hidden symmetries into a *symmetric form* [45]. After transformation

$$r = ix, \quad M = iN_x, \quad N = N_y, \quad (53)$$

the metric and the KY objects take the form

$$ds^2 = \frac{\Delta_x(dt + y^2 d\psi)^2}{x^2 - y^2} + \frac{\Delta_y(dt + x^2 d\psi)^2}{y^2 - x^2} + \frac{(x^2 - y^2)dx^2}{\Delta_x} + \frac{(y^2 - x^2)dy^2}{\Delta_y}, \quad (54)$$

$$\Delta_x = (a^2 - x^2)(1 - \lambda x^2) + 2N_x x, \quad \Delta_y = (a^2 - y^2)(1 - \lambda y^2) + 2N_y y, \quad (55)$$

$$\mathbf{b} = \frac{1}{2}[(x^2 + y^2)d\mathbf{t} + x^2 y^2 d\psi], \quad (56)$$

$$\mathbf{h} = y d\mathbf{y} \wedge (d\mathbf{t} + x^2 d\psi) + x d\mathbf{x} \wedge (d\mathbf{t} + y^2 d\psi). \quad (57)$$

$$\mathbf{f} = x d\mathbf{y} \wedge (d\mathbf{t} + x^2 d\psi) + y d\mathbf{x} \wedge (d\mathbf{t} + y^2 d\psi). \quad (58)$$

This form of the Kerr-NUT-(A)dS spacetime and of the potential (the principal CKY tensor) allows a natural generalization to higher dimensions [45, 58].<sup>12</sup>

### D. Principal conformal Killing–Yano tensor and canonical coordinates

We demonstrate now that the coordinates  $(t, x, y, \psi)$  used in (54)–(58) have a deep invariant meaning. Let us define

$$H^a_b = h^{ac} h_{bc}, \quad \Delta^a_b = H^a_b - H \delta^a_b, \quad (59)$$

then one has

$$\Delta^a_b = \begin{pmatrix} -R^2 - H & aY & -aX & 0 \\ -aY & a^2 - X^2 - H & -YX & -ZX \\ aX & -YX & a^2 - Y^2 - H & -ZY \\ 0 & -ZX & -ZY & -Z^2 - H \end{pmatrix}. \quad (60)$$

---

<sup>12</sup> It is obvious from the derivation that this symmetric form of the metric and of its hidden symmetries is an analytical continuation of the real physical quantities (46), (48), (51). The signature of the metric for this continuation depends on the domain of coordinates  $x$  and  $y$  and the signature of  $\Delta_x$  and  $\Delta_y$ . For example, for  $x > y$  and  $\Delta_x > 0$ ,  $\Delta_y < 0$  it is of the Euclidean signature. The transition to the physical space is given by (53). In higher dimensions it is very convenient to work with a generalization of this symmetric form.



The condition  $\det(\Delta) = 0$  which determines the eigenvalues  $H$  of the operator  $\mathbf{H}^{13}$  is equivalent to the following equation:

$$H^2 + (R^2 - a^2)H - a^2 Z^2 = 0. \quad (62)$$

Hence the eigenvalues of  $\mathbf{H}$  are

$$H_{\pm} = \frac{1}{2} \left[ a^2 - R^2 \pm \sqrt{(R^2 - a^2)^2 + 4a^2 Z^2} \right]. \quad (63)$$

Simple calculations using (43) give

$$H_+ = a^2 \cos^2 \theta = y^2, \quad H_- = -r^2 = x^2. \quad (64)$$

Thus the coordinates  $x$  and  $y$  in (54) are uniquely determined as the eigenvalues of the operator  $\mathbf{H}$  constructed from the principal CKY tensor  $\mathbf{h}$ . Let us show now that the same tensor  $\mathbf{h}$  uniquely determines the coordinates  $t$  and  $\psi$ . The primary Killing vector  $\xi^{(0)}$ , (20), in our case is

$$\xi^{(0)} = \partial_T. \quad (65)$$

Moreover,

$$\xi^{(\psi)a} = -K^{ab} \xi_b^{(0)} = a^2 (\partial_T)^a + aY (\partial_X)^a - aX (\partial_Y)^a, \quad (66)$$

is the *secondary* Killing vector, cf. eq (5). In coordinates (45) one has

$$\xi^{(0)} = \partial_t, \quad \xi^{(\psi)} = \partial_\psi. \quad (67)$$

It means that the coordinates  $t$  and  $\psi$  are the affine parameters along the primary and secondary Killing vectors  $\xi^{(0)}$  and  $\xi^{(\psi)}$ , determined by the tensor  $\mathbf{h}$ . It can be checked that the same is true for the Kerr-NUT-AdS metric (54), (55) with the principal CKY tensor  $\mathbf{h}$  given by (57). This underlines the exceptional role of this tensor. Remarkably, the existence of a similar object in the higher-dimensional Kerr-NUT-(A)dS spacetime generates besides the tower of hidden symmetries also all the isometries of this spacetime, in a way exactly analogous to four dimensions (see Section V). It also determines the canonical coordinates for these metrics [59, 77, 78].

---

<sup>13</sup> The operator  $\mathbf{H}$  is the conformal Killing tensor. It is related to  $\mathbf{K}$  as

$$K_{ab} = H_{ab} - \frac{1}{2} g_{ab} H^c_c, \quad H_{ab} = K_{ab} - \frac{1}{D-2} g_{ab} K^c_c. \quad (61)$$

### E. Separation of variables

The last subject we would like to discuss in this brief review of properties of the 4D Kerr-NUT-(A)dS metrics is the separation of variables for the Hamilton–Jacobi and Klein–Gordon equations.

Let us first consider the Klein–Gordon equation

$$\square\Phi - \mu^2\Phi = 0 \quad (68)$$

in a spacetime (54) with  $\Delta_x$  and  $\Delta_y$  arbitrary functions of  $x$  and  $y$ , respectively. The separation of variables of equation (68) in the canonical coordinates  $(\tau, x, y, \psi)$  means that  $\Phi$  can be decomposed into modes

$$\Phi = e^{i\varepsilon\tau + im\psi} X(x)Y(y). \quad (69)$$

Indeed, substituting this expression in the Klein–Gordon equation (68) one obtains

$$(\Delta_x X')' + V_x X = 0, \quad V_x = \kappa + \mu^2 x^2 - \frac{(\varepsilon x^2 - m)^2}{\Delta_x}, \quad (70)$$

$$(\Delta_y Y')' + V_y Y = 0, \quad V_y = \kappa + \mu^2 y^2 - \frac{(\varepsilon y^2 - m)^2}{\Delta_y}. \quad (71)$$

Here the prime stands for the derivative of function with respect to its single argument. The separation constants  $\varepsilon$  and  $m$  are connected with the symmetries generated by the Killing vectors  $\xi^{(\tau)} = \partial_\tau$  and  $\xi^{(\psi)} = \partial_\psi$ . An additional separation constant  $\kappa$  is connected with the hidden symmetry generated by the Killing tensor  $\mathbf{K}$ . It should be emphasized, that in order to use the proved separability for concrete calculations in the physical Kerr-NUT-(A)dS spacetime (46), (51), one needs to specify functions  $\Delta_x$  and  $\Delta_y$  to have the form (55) and perform the Wick transformation inverse to (53). This transformation ‘spoils’ the symmetry between the essential coordinates but the separability property remains. In coordinates  $r$  and  $y$  in the ‘physical’ sector equations (70) and (71) play different roles. Eq. (71) with imposed regularity conditions serves as an eigenvalue problem which determines the spectrum of  $\kappa$ . Eq. (70) is a radial equation for propagating modes.

Similarly, the Hamilton–Jacobi equation for geodesic motion

$$\partial_\lambda S + g^{ab} \partial_a S \partial_b S = 0 \quad (72)$$

in a generalized metric (54) allows a separation of variables and  $S$  can be written in the form

$$S = \mu^2 \lambda + \varepsilon \tau + m\psi + S_x(x) + S_y(y). \quad (73)$$

The functions  $S_x$  and  $S_y$  obey the equations

$$(S'_x)^2 = \frac{V_x}{\Delta_x}, \quad (S'_y)^2 = \frac{V_y}{\Delta_y}. \quad (74)$$

Presented ‘derivation’ of 4D Kerr-NUT-(A)dS metric and its hidden symmetries can be naturally generalized into higher dimensions. We shall not do this here, but simply mention that the corresponding expressions for the flat spacetime metric in the canonical coordinates and for the potential  $\mathbf{b}$  can be easily obtained by taking the flat space limit of the formulae (90) and (94) (see Section V).

## IV. TOWERS OF KILLING AND KILLING–YANO TENSORS

### A. Important property of closed CKY tensors

The following result [59] plays a central role in the construction of the hidden symmetry objects in higher-dimensional spacetimes.

**Theorem.** *Let  $\mathbf{h}^{(1)}$  and  $\mathbf{h}^{(2)}$  be two closed CKY tensors. Then their external product  $\mathbf{h} = \mathbf{h}^{(1)} \wedge \mathbf{h}^{(2)}$  is also a closed CKY tensor.*

We shall prove this Theorem in two steps. The fact that  $\mathbf{h}$  is closed is trivial, cf. eq. (22). Let us first show that for a  $p$ -form  $\alpha_p$  obeying the equation

$$\nabla_X \alpha_p = \mathbf{X}^\flat \wedge \gamma_{p-1}, \quad (75)$$

one has

$$\gamma_{p-1} = -\frac{1}{D-p+1} \delta \alpha_p. \quad (76)$$

Indeed, we find

$$\begin{aligned} -\delta \alpha_p &= e^a \lrcorner \nabla_a \alpha_p = e^a \lrcorner (\omega_a \wedge \gamma_{p-1}) \\ &= (e^a \lrcorner \omega_a) \gamma_{p-1} - \omega_a \wedge (e^a \lrcorner \gamma_{p-1}) = (D-p+1) \gamma_{p-1}. \end{aligned} \quad (77)$$

Here we have used eq. (25), and relations (27), (28).

The second step in the proof of the Theorem is to show that if  $\alpha_p$  and  $\beta_q$  are two closed CKY tensors then

$$\nabla_X(\alpha_p \wedge \beta_q) = \mathbf{X}^\flat \wedge \gamma_{p+q-1}. \quad (78)$$

Really, one has

$$\begin{aligned} \nabla_X(\alpha_p \wedge \beta_q) &= \nabla_X \alpha_p \wedge \beta_q + \alpha_p \wedge \nabla_X \beta_q \\ &= -\frac{1}{D-p+1}(\mathbf{X}^\flat \wedge \delta \alpha_p) \wedge \beta_q - \frac{1}{D-q+1} \alpha_p \wedge (\mathbf{X}^\flat \wedge \delta \beta_q) \\ &= \mathbf{X}^\flat \wedge \gamma_{p+q-1}, \end{aligned} \quad (79)$$

where

$$\gamma_{p+q-1} = -\frac{1}{D-p+1} \delta \alpha_p \wedge \beta_q - \frac{(-1)^p}{D-q+1} \alpha_p \wedge \delta \beta_q. \quad (80)$$

Combining (79) with (76) we arrive at the statement of the theorem.

## B. Principal CKY tensor and towers of hidden symmetries

Let us consider now a special case which is important for applications. Namely, we assume that a spacetime allows a 2-form  $\mathbf{h}$  which is a *closed CKY tensor*,  $\mathbf{h} = d\mathbf{b}$ . We also assume that  $\mathbf{h}$  is *non-degenerate*, that is its (matrix) rank is  $2n$ . We call this object a *principal CKY tensor* [59].

According to the Theorem of the previous section, the principal CKY tensor generates a set (tower) of new closed CKY tensors

$$\mathbf{h}^{(j)} = \mathbf{h}^{\wedge j} = \underbrace{\mathbf{h} \wedge \dots \wedge \mathbf{h}}_{\text{total of } j \text{ factors}}. \quad (81)$$

$\mathbf{h}^{(j)}$  is a  $(2j)$ -form, and in particular  $\mathbf{h}^{(1)} = \mathbf{h}$ . Since  $\mathbf{h}$  is non-degenerate, one has a set of  $n$  non-vanishing closed CKY tensors. In an even dimensional spacetime  $\mathbf{h}^{(n)}$  is proportional to the totally antisymmetric tensor whereas it is dual to a Killing vector in odd dimensions. In both cases such CKY tensor is trivial and can be excluded from the tower of hidden symmetries. The CKY tensors can be generated from the potentials  $\mathbf{b}^{(j)}$  (cf. eq. (32))

$$\mathbf{b}^{(j)} = \mathbf{b} \wedge \mathbf{h}^{\wedge(j-1)}, \quad \mathbf{h}^{(j)} = d\mathbf{b}^{(j)}. \quad (82)$$

Each  $(2j)$ -form  $\mathbf{h}^{(j)}$  determines a  $(D-2j)$ -form of the Killing–Yano tensor

$$\mathbf{f}^{(j)} = *\mathbf{h}^{(j)}. \quad (83)$$

In their turn, these tensors give rise to the Killing tensors  $\mathbf{K}^{(j)}$

$$K_{ab}^{(j)} = \frac{1}{(D-2j-1)!(j!)^2} f_{ac_1 \dots c_{D-2j-1}}^{(j)} f_b^{(j) c_1 \dots c_{D-2j-1}}. \quad (84)$$

A choice of the coefficient in the definition (84) is adjusted to Section V, cf. eq. (19). It is also convenient to include the metric  $\mathbf{g}$ , which is a trivial Killing tensor, as an element  $\mathbf{K}^{(0)}$  of the tower of the Killing tensors. The total number of irreducible elements of this *extended tower* is  $n$ .<sup>14</sup>

### C. Canonical basis and canonical coordinates

Similar to 4D case, let us consider the eigenvalue problem for the conformal Killing tensor  $H_{ab} = h_{ac}h_b{}^c$ . It is easy to show that in the Euclidean domain its eigenvalues  $x^2$ ,

$$H_b^a v^b = x^2 v^a, \quad (85)$$

are real and non-negative. Using a modified Gram–Schmidt procedure it is possible to show that there exists such an orthonormal basis in which the operator  $\mathbf{h}$  has the following structure:

$$\text{diag}(0, \dots, 0, \Lambda_1, \dots, \Lambda_p), \quad (86)$$

where  $\Lambda_i$  are matrices of the form

$$\Lambda_i = \begin{pmatrix} 0 & -x_i I_i \\ x_i I_i & 0 \end{pmatrix}, \quad (87)$$

and  $I_i$  are unit matrices. Such a basis is known as the *Darboux basis* (see, e.g., [79]). Its elements are unit eigenvectors of the problem (85).

For a non-degenerate 2-form  $\mathbf{h}$  the number of zeros in the Darboux decomposition (86) coincides with  $\varepsilon$ . If all the eigenvalues  $x$  in (85) are different (we denote them  $x_\mu$ ,  $\mu = 1, \dots, n$ ), the matrices  $\Lambda_i$  are 2-dimensional. Denote the vectors of the Darboux basis by  $\mathbf{e}_\mu$  and  $\mathbf{e}_{\bar{\mu}} \equiv \mathbf{e}_{n+\mu}$ , where  $\mu = 1, \dots, n$ . In an odd dimensional spacetime we also have an additional basis vector  $\mathbf{e}_0$  (the eigenvector of (85) with  $x = 0$ ). Orthonormal vectors  $\mathbf{e}_\mu$

---

<sup>14</sup> For example, in 5D spacetime where  $n = 2$ , this tower contains only one non-trivial Killing tensor. For the 5D rotating black hole solution this Killing tensor was first found in [52, 53] by using the Carter's method of separation of variables in the Hamilton–Jacobi equation.

and  $e_{\bar{\mu}}$  span a 2-dimensional plane of eigenvectors of (85) with the same eigenvalue  $x_{\mu}$ . We denote by  $\omega^{\mu}$  and  $\omega^{\bar{\mu}} \equiv \omega^{n+\mu}$  (and  $\omega^0$  if  $\varepsilon = 1$ ) the dual basis of 1-forms. The metric  $g$  and the principal CKY tensor  $h$  in this basis take the form

$$g = \sum_{\mu=1}^n (\omega^{\mu} \omega^{\mu} + \omega^{\bar{\mu}} \omega^{\bar{\mu}}) + \varepsilon \omega^0 \omega^0, \quad (88)$$

$$h = \sum_{\mu=1}^n x_{\mu} \omega^{\mu} \wedge \omega^{\bar{\mu}}. \quad (89)$$

## V. HIDDEN SYMMETRIES OF HIGHER-DIMENSIONAL KERR-NUT-(A)DS SPACETIMES

The most general known higher-dimensional solution describing rotating black holes with NUT parameters in an asymptotically (Anti) de Sitter spacetime (Kerr-NUT-(A)dS metric) was found by Chen, Lü, and Pope [45]. In an (analytically continued) symmetric form it is written as (88) where

$$\omega^{\mu} = \frac{dx_{\mu}}{\sqrt{Q_{\mu}}}, \quad \omega^{\bar{\mu}} = \sqrt{Q_{\mu}} \sum_{k=0}^{n-1} A_{\mu}^{(k)} d\psi_k, \quad \omega^0 = (c/A^{(n)})^{1/2} \sum_{k=0}^n A^{(k)} d\psi_k. \quad (90)$$

Here

$$Q_{\mu} = X_{\mu}/U_{\mu}, \quad U_{\mu} = \prod_{\nu \neq \mu} (x_{\nu}^2 - x_{\mu}^2),$$

$$A_{\mu}^{(k)} = \sum_{\substack{\nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2, \quad A^{(k)} = \sum_{\nu_1 < \dots < \nu_k} x_{\nu_1}^2 \dots x_{\nu_k}^2. \quad (91)$$

Metric coefficients  $X_{\mu}$  are functions of  $x_{\mu}$  only, and for the Kerr-NUT-(A)dS solution take the form

$$X_{\mu} = \sum_{k=\varepsilon}^n c_k x_{\mu}^{2k} - 2b_{\mu} x_{\mu}^{1-\varepsilon} - \frac{\varepsilon c}{x_{\mu}^2}. \quad (92)$$

Time is denoted by  $\psi_0$ , azimuthal coordinates by  $\psi_k$ ,  $k = 1, \dots, m$ , and  $x_{\mu}$ ,  $\mu = 1, \dots, n$ , stand for ‘radial’ and latitude coordinates. Here we have introduced  $m = n - 1 + \varepsilon$ . The physical metric with proper signature is recovered when standard radial coordinate  $r = -ix_n$  and new parameter  $M = (-i)^{1+\varepsilon} b_n$  are introduced. The total number of constants which enter the solution is  $2n + 1$ :  $\varepsilon$  constants  $c$ ,  $n + 1 - \varepsilon$  constants  $c_k$  and  $n$  constants  $b_{\mu}$ . The form of the metric is invariant under a 1-parameter scaling coordinate transformations, thus

a total number of independent parameters is  $D - \varepsilon$ . These parameters are related to the cosmological constant, mass, angular momenta, and NUT parameters. One of them may be used to define a scale, while the other  $D - 1 - \varepsilon$  parameters can be made dimensionless. (For more details see [45].) Similar to the 4D case, the signature of the symmetric form of the metric depends on the domain of  $x_\mu$ 's and the signatures of  $X_\mu$ 's.

Hamamoto, Houri, Oota, and Yasui [80] derived explicit formulas for the curvature and demonstrated that in all dimensions this metric obeys the Einstein equations,

$$R_{ab} = \Lambda g_{ab}, \quad \Lambda = (D - 1)(-1)^n c_n. \quad (93)$$

The limit of flat spacetime is recovered when  $c_n = 0$  and all of the parameters  $b_\mu$  are zero (equal one to each other) in the even (odd) dimensional case. The metric belongs to the class of special algebraic type **D** solutions [80] of the higher-dimensional algebraic classification [81]. It may be understood as a higher-dimensional generalization of the 4-dimensional Kerr-NUT-(A)dS solution obtained by Carter [8]. Moreover, the coordinates  $(x_\mu, \psi_k)$  used in the metric are the higher-dimensional analogue of the canonical coordinates [8, 32].

It was shown in [57, 58] that this spacetime possesses a principal CKY tensor  $\mathbf{h}$  which has the form (89) and its potential  $\mathbf{b}$ ,  $\mathbf{h} = d\mathbf{b}$ , is<sup>15</sup>

$$\mathbf{b} = \frac{1}{2} \sum_{k=0}^{n-1} A^{(k+1)} d\psi_k. \quad (94)$$

The tower of Killing tensors (84) associated with this principal CKY tensor is [59]

$$\mathbf{K}^{(j)} = \sum_{\mu=1}^n A_\mu^{(j)} (\boldsymbol{\omega}^\mu \boldsymbol{\omega}^\mu + \boldsymbol{\omega}^{\bar{\mu}} \boldsymbol{\omega}^{\bar{\mu}}) + \varepsilon A^{(j)} \boldsymbol{\omega}^0 \boldsymbol{\omega}^0, \quad j = 1, \dots, n-1. \quad (95)$$

In the Kerr-NUT-(A)dS spacetime besides this tower of Killing tensors also all the isometries follow from the existence of the principal CKY tensor  $\mathbf{h}$  [59]. The *primary Killing vector*  $\boldsymbol{\xi}^{(0)}$  is given by (20) and reads

$$\boldsymbol{\xi}^{(0)} = \partial_{\psi_0}. \quad (96)$$

This vector plays a special role. After the analytical continuation to the physical space it is the only one timelike in the black hole exterior. Besides the primary Killing vector, this

---

<sup>15</sup> In fact, this potential generates a principal CKY tensor for a general form of the metric (88),(90), and (91) with arbitrary functions  $X_\mu(x_\mu)$ .

spacetime has  $n - 1$  *secondary Killing vectors*  $\xi^{(j)}$ ,

$$\xi^{(j)a} = K^{(j)a}{}_b \xi^b = (\partial_{\psi_j})^a, \quad j = 1, \dots, n - 1. \quad (97)$$

In odd dimensions the last Killing vector is given by the  $n$ -th Killing–Yano tensor  $\mathbf{f}^{(n)}$ , which in the Kerr-NUT-(A)dS spacetime turns out to be

$$\xi^{(n)} = \mathbf{f}^{(n)} = \partial_{\psi_n}. \quad (98)$$

The total number of these Killing vectors is  $n + \varepsilon$ . For a geodesic motion they give  $n + \varepsilon$  linear in momentum integrals of motion,  $\Psi_j = \xi_a^{(j)} p^a$ . The extended tower of the Killing tensors  $\mathbf{K}^{(j)}$  ( $j = 0, \dots, n - 1$ ) gives  $n$  additional integrals of motion, which are quadratic in the momentum,  $\kappa_j = K_{ab}^{(j)} p^a p^b$ . Thus the total number of conserved quantities for a geodesic motion is  $2n + \varepsilon$ , that is it coincides with the number of spacetime dimensions  $D$ . It is possible to show that these integrals of motion are independent and in involution, so that the geodesic motion in the Kerr-NUT-(A)dS spacetime is completely integrable [60, 61]. The components of the particle momentum  $\mathbf{p}$  can be written as functions of the integrals of motion as follows [59]:

$$\begin{aligned} p_\mu &= \frac{\sigma_\mu}{(X_\mu U_\mu)^{1/2}} \left[ X_\mu \sum_{j=0}^m (-x_\mu^2)^{n-1-j} \kappa_j - \left( \sum_{j=0}^m (-x_\mu^2)^{n-1-j} \Psi_j \right)^2 \right]^{1/2}, \\ p_{\bar{\mu}} &= \frac{1}{(X_\mu U_\mu)^{1/2}} \sum_{j=0}^m (-x_\mu^2)^{n-1-j} \Psi_j, \\ p_0 &= (cA^{(n)})^{-1/2} \Psi_n, \quad \kappa_n = \frac{\Psi_n^2}{c}. \end{aligned} \quad (99)$$

Constants  $\sigma_\mu$  which denote the choice of signs are independent of one another.

It should be emphasized that, similar to 4D case, the coordinates  $(x_\mu, \psi_k)$  in the metric (88), (90), (91) have a well defined geometrical meaning. The ‘essential’ coordinates  $x_\mu$  are connected with the eigenvalues of the principal CKY tensor  $\mathbf{h}$  (see (89)), while the Killing coordinates  $\psi_k$  are defined by the Killing vectors generated by the principal CKY tensor. It is this invariant definition of the coordinates what makes this form of the metric so convenient for calculations.

The existence of a principal CKY tensor imposes non-trivial restrictions on the geometry of the spacetime. Namely, the following result was proved in [77, 78]. Let  $\mathbf{h}$  be a principal CKY tensor and  $\xi^{(0)}$  be its primary Killing vector. Then if

$$\mathcal{L}_{\xi^{(0)}} \mathbf{h} = 0, \quad (100)$$



the only solution of the higher-dimensional Einstein equations with the cosmological constant (93) is the Kerr-NUT-(A)dS spacetime. (Here  $\mathcal{L}_u$  is a Lie derivative along the vector  $u$ .)

Let us finally mention that recently it was shown [82] that the following operators (cf. eq. (6)):

$$\hat{\xi}_{(k)} = -i\xi^{(k)a}\partial_a, \quad k = 0, \dots, m, \quad (101)$$

$$\hat{K}_{(j)} = -\frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}K^{(j)ab}\partial_b\right), \quad j = 0, \dots, n-1, \quad (102)$$

determined by a principal CKY tensor, form a complete set of commuting operators for the Klein-Gordon equation in the Kerr-NUT-(A)dS background.

## VI. HIDDEN SYMMETRIES AND SEPARATION OF VARIABLES

The massive scalar field equation

$$\square\Phi - \mu^2\Phi = 0, \quad (103)$$

in the Kerr-NUT-(A)dS metric allows a complete separation of variables [62]. Namely, the solution can be decomposed into modes

$$\Phi = \prod_{\mu=1}^n R_\mu(x_\mu) \prod_{k=0}^m e^{i\Psi_k\psi_k}. \quad (104)$$

Substitution of (104) into equation (103) results in the following second order ordinary differential equations for functions  $R_\mu(x_\mu)$

$$(X_\mu R'_\mu)' + \varepsilon \frac{X_\mu}{x_\mu} R'_\mu + \left(V_\mu - \frac{W_\mu^2}{X_\mu}\right) R_\mu = 0. \quad (105)$$

Here

$$W_\mu = \sum_{k=0}^m \Psi_k (-x_\mu^2)^{n-1-k}, \quad V_\mu = \sum_{k=0}^m \kappa_k (-x_\mu^2)^{n-1-k}. \quad (106)$$

Here  $\kappa_0 = -\mu^2$  and for  $\varepsilon = 1$  we put  $\kappa_n = \Psi_n^2/c$ . The parameters  $\kappa_k$  ( $k = 1, \dots, n + \varepsilon - 1$ ) are separation constants. Using (102) one has  $\hat{K}_{(0)} = -\square$ . Since all the operators (101)–(102) commute with one another, their common eigenvalues can be used to specify the modes. It is possible to show [82] that the eigenvectors of these commuting operators are the modes (104) and one has

$$\hat{\xi}_{(k)}\Phi = \Psi_k\Phi, \quad \hat{K}_{(j)}\Phi = \kappa_j\Phi. \quad (107)$$

Similar to the case discussed in 4D, in the symmetric form of the metric (88), (90), (91) all the equations (105) ‘look the same’. However, after the transformation to the physical space the equation for  $R_n$  plays the role of an equation for radial modes, whereas the other equations present the eigenvalue problem. For a discussion of special sub-cases of these equations see, e.g, [83] and reference therein.

The Hamilton–Jacobi equation for geodesic motion

$$\frac{\partial S}{\partial \lambda} + g^{ab} \partial_a S \partial_b S = 0, \quad (108)$$

in the Kerr-NUT-(A)dS spacetime also allows a complete separation of variables [62]

$$S = \mu^2 \lambda + \sum_{k=0}^m \Psi_k \psi_k + \sum_{\mu=1}^n S_\mu(x_\mu). \quad (109)$$

The functions  $S_\mu$  obey the first order ordinary differential equations

$$S'_\mu{}^2 = \frac{V_\mu}{X_\mu} - \frac{W_\mu^2}{X_\mu^2}, \quad (110)$$

where the functions  $V_\mu$  and  $W_\mu$  are defined in (106).

Recently it was shown that the massive Dirac equation in the Kerr-NUT-(A)dS spacetime also allows the separation of variables [63]. It was also proved that the Nambu-Goto equations for a stationary test string in the Kerr-NUT-(A)dS background are completely integrable [65].

## VII. CONCLUSIONS

In this review we discussed recent developments of the theory of higher-dimensional black holes. We focused mainly on the problem of hidden symmetries and separation of variables. (For more general discussion of the modern status of the theory of higher-dimensional black holes see a recent review [51].) In our presentation, we started with a description of known results concerning four-dimensional black holes. We found this important since many of the properties of higher-dimensional isolated black holes are quite similar to the properties of black holes in four dimensions. What we tried to illustrate in this presentation is that most of the important properties of a stationary isolated black hole solution follow from the existence of what is called a *principal conformal Killing–Yano tensor*. This is true in four dimensions and, as recent studies demonstrated, in higher dimensions as well.

The Kerr-NUT-(A)dS metric is the most general known solution describing the higher-dimensional rotating black hole spacetime with NUT parameters in an asymptotically (Anti) de Sitter background. It possesses a principal CKY tensor  $\mathbf{h}$  which determines the hidden symmetries of this spacetime. The 2-form  $\mathbf{h}$  generates a tower of Killing–Yano and Killing tensors, which allow complete integrability of geodesic equations and separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations. Moreover, if the higher-dimensional solution of the Einstein equations with the cosmological constant (often called the *Einstein space*) allows a principal CKY tensor obeying (100), it coincides with the Kerr-NUT-(A)dS metric [80]. These remarkable properties of higher-dimensional rotating black holes resemble the well known miraculous properties of the Kerr spacetime described partly in the introduction. In four dimensions all of the Einstein spaces which possess the KY tensor are of the Petrov type **D**. A natural conjecture in higher dimensions is that any Einstein space which possesses the principal CKY tensor is of the special algebraic type **D** of higher-dimensional classification [81].

Does this analogy go further? We focused on black holes, but in the higher-dimensional gravity there exists a variety of other black objects such as black rings or black saturns. Do these spacetimes also possess hidden symmetries? Such symmetries would be very helpful for the study of the stability of these ‘exotic’ objects. However, for example black rings are of the algebraic type  $\mathbf{I}_i$  [86] and thus their symmetry properties might be quite different from the properties of higher-dimensional black holes. In particular, if the above conjecture is correct they do not allow a principle CKY tensor and so nor a corresponding tower of hidden symmetries.

The results on the separability till now obtained can be used for the study of the particle and light propagation in higher-dimensional rotating black hole spacetimes. They allow one to calculate the contribution of the scalar and Dirac fields to the bulk Hawking radiation of higher-dimensional rotating black holes, without any restrictions on black hole parameters.

An important open question is a separability problem for the gravitational perturbations in higher-dimensional black hole spacetimes. A certain progress in this direction was achieved recently (see, e.g., [84], [64]). These results are very important for the study of the stability of such black holes and different aspect of the Hawking radiation produced by them. Another important direction of research is the study of the quasinormal modes in higher dimensions. Most of the results obtained in these directions (see, e.g., [85] and references therein) assumed

some additional restrictions on the parameters characterizing black hole solutions. This remind a situation for the Klein–Gordon and Dirac equations before the general results on their separability were proved.

An important open question is: Can the higher spin massless field equations be decoupled in the background of the general Kerr-NUT-(A)dS metric and do they allow separation of variables? Recent result of [63] on the separability of the massive Dirac equation is quite promising. Separability of the higher spin equations, and especially the equations for the gravitational perturbations, would provide one with powerful tools important, for example, for the study of the stability of higher-dimensional black hole solutions. One might hope that it will not take too long time before this and other interesting open questions connected with the existence of hidden symmetries in higher-dimensional black holes will find their answers.

### Acknowledgements

This paper is an extended version of the talk given by one of the authors, V.F., at the “Peyresq Physics 12” meeting on “Micro and Macro structure of spacetime” held in Peyresq in June 18-24, 2007. He appreciate very much stimulating scientific atmosphere of Peyresq village created by Mady Smets, who devoted this place to humanistic activities. V.F thanks “OLAM Association for Fundamental Research, Brussels, Belgium” for its support, and Diane and Edgard Gunzig for their hospitality. V.F. is also grateful to the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for partial support. D.K. acknowledges the Golden Bell Jar Graduate Scholarship in Physics at the University of Alberta.

- 
- [1] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford)
  - [2] Kerr R P 1963 *Phys. Rev. Lett.* **11** 237
  - [3] Kerr R P and Schild A 1965 *Atti Del Convegno Relativita Generale: Problemi Dell Energia E Onde Gravitazionali*, edited by Barbera G (Firenze, Italy) 173; Kerr R P and Schild A 1969 *Proc. Symp. Appl. Math.* **17** 199; Debney G C, Kerr R P and Schild A 1969 *J. Math. Phys.* **10** 1842
  - [4] Gürses M and Gürsey F 1975 *J. Math. Phys.* **16** 2385

- [5] Myers R C and Perry M J 1986 *Ann. Phys. (N.Y.)* **172** 304
- [6] Petrov A Z 1954 *Reports of the State University of Kazan* **114**, book 8, 55 ; Petrov A Z 1969 *Einstein spaces*, New York: Pergamon
- [7] Carter B 1968 *Phys. Rev.* **174** 1559
- [8] Carter B 1968 *Commun. Math. Phys.* **10** 280
- [9] Walker M and Penrose R 1970 *Commun. Math. Phys.* **18** 265
- [10] Stackel P 1895 *C. R. Acad. Sci. Paris* **121** 489
- [11] Teukolsky S A 1972 *Phys. Rev. Lett.* **29** 1114
- [12] Unruh W 1973 *Phys. Rev. Lett.* **31** 1265
- [13] Teukolsky S A 1973 *Astrophys. J.* **185** 635
- [14] Chandrasekhar S 1976 *Proc. Roy. Soc. London A* **349** 571
- [15] Page D N 1976 *Phys. Rev. D* **14** 1509
- [16] Penrose R 1973 *Ann. N. Y. Acad. Sci.* **224** 125 Floyd R 1973 *The dynamics of Kerr fields*, PhD Thesis, London
- [17] Yano K 1952 *Ann. Math.* **55** 328
- [18] Collinson C D 1976 *Int. J. Theor. Phys.* **15** 311; Stephani H 1977 *Gen. Rel. Grav.* **9** 789; Ferrando J J and Sáez J A 2002 *Gen. Rel. Grav.* **35** 1191
- [19] Collinson C D 1974 *Tensor* **28** 173
- [20] Kinnersley W 1969 *J. Math. Phys.* **10** 1195
- [21] Demianski M and Francaviglia M 1980 *Int. J. Theor. Phys.* **19** 675
- [22] Hughston L P and Sommers P 1973 *Comm. Math. Phys.* **33** 129
- [23] Carter B 1977 *Phys. Rev. D* **16** 3395
- [24] Cariglia M 2004 *Class. Quantum Grav.* **21** 1051
- [25] Carter B and McLenaghan R G 1979 *Phys. Rev. D* **19** 1093
- [26] Kamran N and McLenaghan R.G 1984 *Phys. Rev. D* **30** 357; Kamran N 1985 *J. Math. Phys.* **26** 1740; Torres del Castillo G F 1988 *J. Math. Phys.* **29** 971; Kalnis E G, Miller Jr. W and Williams G C 1989 *J. Math. Phys.* **30** 2630; Kalnis E G, Williams G C and Miller Jr. W 1996 *Proc. R. Soc. A* **452** 997
- [27] Marck J A 1983 *Proc. R. Soc. A* **385** 431
- [28] Carter B 1987 *J. Math. Phys.* **28** 1535
- [29] Frolov V P, Skarzhinsky V D, Zel'nikov A I and Heinrich O 1989 *Phys. Lett. B* **224** 255;

- Carter B and Frolov V P 1989 *Class. Quantum Grav.* **6** 569; Carter B, Frolov V P and Heinrich O 1991 *Class. Quantum Grav.* **8** 135
- [30] Gibbons G W, Rietdijk R H and van Holten J W 1993 *Nucl. Phys. B* **404** 42
- [31] Jezierski J and Lukasik M 2006 *Class. Quantum Grav.* **23** 2895
- [32] Carter B 1968 *Phys. Lett. A* **26** 399; see also, Debever R *Bull. Soc. Math. Belg.* 1971 **23** 360; Plebański J F 1975 *Ann. Phys., NY* **90** 196
- [33] Dietz W and Rüdiger R 1981 *Proc. R. Soc. A* **375** 361; Dietz W and Rüdiger R 1982 *Proc. R. Soc. A* **381** 315
- [34] Hall G S 1987 *Int. J. Theor. Phys.* **26** 71
- [35] Woodhouse N M J 1975 *Comm. Math. Phys.* **44** 9; Benenti S and Francaviglia M 1979 *Gen. Rel. Grav.* **10** 79; Kalnis E G and Miller Jr. W 1981 *SIAM J. Math. Anal.* **12** 617
- [36] Banks T and Fischler W 1999 A model for high energy scattering in quantum gravity *Preprint* hep-th/9906038; Emparan R, Horowitz G T and Myers R C 2000 *Phys. Rev. Lett.* **85** 499; Dimopoulos S and Landsberg G L 2001 *Phys. Rev. Lett.* **87** 161602; Giddings S B and Thomas S D 2002 *Phys. Rev. D* **65** 056010; Eardley D M and Giddings S B 2002 *Phys. Rev. D* **66** 044011; Yoshino H and Nambu Y 2003 *Phys. Rev. D* **67** 024009; Kanti P 2004 *Int. J. Mod. Phys. A* **19** 4899; Harris C M, Kanti P 2006 *Phys. Lett. B* **633** 106; Kanti P 2007 *AIP Conf. Proc.* **903** 467; Giddings S B 2007 *AIP Conf. Proc.* **957** 69; Dvali G and Sibiryakov S 2008 *J. High Energy Phys.* JHEP03(2008)007
- [37] Tangherlini F R 1963 *Nuovo Cimento* **27** 636
- [38] Schwarzschild K 1916 *Sitzungsber. deutsch. Akad. Wiss. Berlin*, Kl. Math. Phys. Technik 189
- [39] Hawking S W, Hunter C J and Taylor-Robinson M M 1999 *Phys. Rev. D* **59** 064005
- [40] Gibbons G W, Lü H, Page D N and Pope C N 2005 *J. Geom. Phys.* **53** 49
- [41] Gibbons G W, Lü H, Page D N and Pope C N 2004 *Phys. Rev. Lett.* **93** 171102
- [42] Newman E, Tamburino L and Unti T 1963 *J. Math. Phys. (N.Y.)* **4** 915
- [43] Chong Z-W, Gibbons G W, Lü H and Pope C N 2005 *Phys. Lett B* **609** 124
- [44] Chen W, Lü H and Pope C N 2007 *Nucl. Phys. B* **762** 38
- [45] Chen W, Lü H and Pope C N 2006 *Class. Quantum Grav.* **23** 5323
- [46] Hashimoto N, Sakaguchi M and Yasui Y 2004 *Phys. Lett. B* **600** 270; Cvetič M, Lü H, Page D N and Pope C N 2005 *Phys. Rev. Lett.* **95** 071101; Cvetič M, Lü H, Page D N and Pope C N 2005 New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter *Preprint* hep-th/0505223

- [47] Oota T and Yasui Y 2006 *Phys. Lett. B* **639** 54; Lü H and Pope C N 2007 *Nucl. Phys. B* **782** 171
- [48] Aliev A N and Frolov V P 2004 *Phys. Rev. D* **69** 084022; Aliev A N 2006 *Phys. Rev. D* **74** 024011; Aliev A N 2007 *Phys. Rev. D* **75** 084041; Kunz J, Maison D, Navarro-Lenda F and Viebahn J 2006 *Phys. Lett. B* **639** 95; Chen W and Lü H 2008 *Phys. Lett. B* **658** 158; Krtouš P 2007 *Phys. Rev. D* **76** 084035
- [49] Kunz J, Navarro-Lérida F and Peterson A K 2005 *Phys. Lett. B* **614** 104; Kunz J, Navarro-Lérida F and Viebahn J 2006 *Phys. Lett. B* **639** 362; Kunz J, Navarro-Lérida F and Radu E 2007 *Phys. Lett. B* **649** 463; Brihaye Y and Delsate T 2007 *Class. Quantum Grav.* **24** 4691; Kleihaus B, Kunz J and Navarro-Lérida F 2008 *AIP Conf. Proc.* **977** 94
- [50] Charmousis C and Gregory R 2004 *Class. Quantum Grav.* **21** 527; Podolský J and Ortaggio M 2006 *Class. Quantum Grav.* **23** 5785; Pravda V, Pravdová A and Ortaggio M 2007 *Class. Quantum Grav.* **24** 4407; Ortaggio M, Podolský J and Žofka M 2008 *Class. Quantum Grav.* **25** 025006
- [51] Emparan R and Reall H S 2008 Black Holes in Higher Dimensions *Preprint* arXiv:0801.3471
- [52] Frolov V P and Stojkovic D 2003 *Phys. Rev. D* **67** 084004
- [53] Frolov V P and Stojkovic D 2003 *Phys. Rev. D* **68** 064011
- [54] Gooding C and Frolov A V 2008 Five-dimensional black hole capture cross-sections *Preprint* arXiv:0803.1031
- [55] Ortega A L 2003 *Gen. Rel. Grav.* **35** 1785; Vasudevan M, Stevens K A and Page D N 2005 *Class. Quantum Grav.* **22** 1469; Kunduri H K and Lucietti J 2005 *Phys. Rev. D* **71** 104021; Vasudevan M, Stevens K A and Page D N 2005 *Class. Quantum Grav.* **22** 339; Vasudevan M and Stevens K A 2005 *Phys. Rev. D* **72** 124008; Vasudevan M 2006 *Phys. Lett. B* **632** 532; Chen W, Lu H and Pope C N 2006 *J. High Energy Phys.* JHEP04(2006)008; Davis P 2006 *Class. Quantum Grav.* **23** 3607; Vasudevan M 2006 *Symmetries of Black Holes and D-Branes*, Ph.D. Thesis, University of Alberta
- [56] Frolov V P and Stevens K A 2004 *Phys. Rev. D* **70** 44035
- [57] Frolov V P and Kubizňák D 2007 *Phys. Rev. Lett.* **98** 011101
- [58] Kubizňák D and Frolov V P 2007 *Class. Quantum Grav.* **24** F1
- [59] Krtouš P, Kubizňák D, Page D N and Frolov V P 2007 *J. High Energy Phys.* JHEP02(2007)004
- [60] Page D N, Kubizňák D, Vasudevan M and Krtouš P 2007 *Phys. Rev. Lett.* **98** 061102

- [61] Krtouš P, Kubizňák D, Page D N and Vasudevan M 2007 *Phys. Rev. D* **76** 084034
- [62] Frolov V P, Krtouš P and Kubizňák D 2007 *J. High Energy Phys.* JHEP02(2007)005
- [63] Oota T and Yasui Y 2008 *Phys. Lett. B* **659** 688
- [64] Murata K and Soda J 2008 *Class. Quantum Grav.* **25** 035006
- [65] Kubizňák D and Frolov V P 2008 *J. High Energy Phys.* JHEP02(2008)007
- [66] Frolov V P 2007 Hidden Symmetries of Higher-Dimensional Black Hole Spacetimes *Preprint* arXiv:0712.4157
- [67] Tachibana S 1969 *Tôhoku Math. J.* **21** 56; Kashiwada T 1968 *Nat. Sci. Rep. Ochanomizu University* **19** 67
- [68] Jezierski J 1997 *Class. Quantum Grav.* **14** 1679
- [69] Sternberg S 1964 *Lectures on Differential Geometry*, Prentice Hall, Inc. Englewood Cliffs, N.Y.
- [70] Benn I M, Charlton P and Kress J 1997 *J. Math. Phys.* **38** 4504; Benn I M and Charleton P 1997 *Class. Quantum Grav.* **14** 1037; Kress J 1997 *Generalized Conformal Killing-Yano Tensors: Applications to Electrodynamics*, PhD Thesis, University of Newcastle; Charleton P 1997 *The Geometry of Pure Spinors, with Applications*, PhD Thesis, University of Newcastle
- [71] Dadhich N and Turakulov Z Y 2002 *Class. Quantum Grav.* **19** 2765
- [72] Didenko V E, Matveev A S and Vasiliev M A 2008 Unfolded Description of AdS(4) Kerr Black Hole *Preprint* arXiv:0801.2213
- [73] Kastor D and Traschen J 2004 *J. High Energy Phys.* JHEP08(2004)045
- [74] Griffiths J B and Podolský J 2007 *Class. Quantum Grav.* **24** 1687
- [75] Kubizňák D and Krtouš P 2007 *Phys. Rev. D* **76** 084036
- [76] Rosquist K, Bylund T and Samuelsson L 2007 Carter's constant revealed *Preprint* arXiv:0710.4260
- [77] Houri T, Oota T and Yasui Y 2008 *J. Phys. A* **41** 025204.
- [78] Houri T, Oota T and Yasui Y 2007 *Phys. Lett. B* **656** 214
- [79] Prasolov V V 1994 *Problems and Theorems in Linear Algebra*, Translations of Mathematical monographs, vol. 134, American Math. Society, Providence, Rhode Island
- [80] Hamamoto N, Houri T, Oota T and Yasui Y 2007 *J. Phys. A* **40** F177
- [81] Milson R, Coley A, Pravda V and Pravdová A 2005 *Int. J. Geom. Meth. Mod. Phys.* **2** 41; Coley A, Milson R, Pravda V and Pravdová A 2004 *Class. Quantum Grav.* **21** L35; Coley A 2008 *Class. Quantum Grav.* **25** 033001



- [82] Sergyeyev A and Krtouš P 2008 *Phys. Rev. D* **77** 044033
- [83] Bert E, Cardoso V and Casals M 2006 *Phys. Rev. D* **73** 024013
- [84] Kodama H and Ishibashi A 2003 *Prog. Theor. Phys.* **110** 701; Kunduri H K, Lucietti J and Reall H S 2006 *Phys. Rev. D* **74** 084021
- [85] Ida D, Uchida Y and Morisawa Y 2003 *Phys. Rev. D* **67** 084019; Cardoso V, Dias J C and Lemos J P S 2003 *Phys. Rev. D* **67** 064026; Cardoso V, Lemos J P S and Yoshida S 2004 *Phys. Rev. D* **69** 044004; Konoplya R A 2003 *Phys. Rev. D* **68** 024018; Cardoso V, Cavaglia M and Gualtieri L 2006 *Phys. Rev. Lett.* **96** 071301; Zhidenko Z 2006 *Phys. Rev. D* **74** 064017; Kanti P, Konoplya R A and Zhidenko A 2006 *Phys. Rev. D* **74** 064008; Lopez-Ortega A 2006 *Gen. Rel. Grav.* **38** 1565; Lopez-Ortega A 2006 *Gen. Rel. Grav.* **38** 1747; Lopez-Ortega A 2007 *Gen. Rel. Grav.* **39** 1011; Konoplya R A and Zhidenko A 2007 *Nucl. Phys. B* **777** 182; Lopez-Ortega A 2007 Electromagnetic quasinormal modes of D-dimensional black holes II *Preprint* arXiv:0706.2933; Kodama H 2007 Superradiance and Instability of Black Holes *Preprint* arXiv:0711.4184; Casals M, Dolan S R, Kanti P and Winstanley E 2008 Bulk Emission of Scalars by a Rotating Black Hole *Preprint* arXiv:0801.4910
- [86] Pravda V and Pravdová A 2005 *Class. Quantum Grav.* **37** 1277